

# Generalised monogamy relation of convex-roof extended negativity in multi-level systems

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## ABSTRACT

In this paper, we investigate the generalised monogamy inequalities of convex-roof extended negativity (CREN) in multi-level systems. The generalised monogamy inequalities provide the upper and lower bounds of bipartite entanglement, which are obtained by using CREN and the CREN of assistance (CRENOA). Furthermore, we show that the CREN of multi-qubit pure states satisfies some monogamy relations. Additionally, we test the generalised monogamy inequalities for qudits by considering the partially coherent superposition of a generalised W-class state in a vacuum and show that the generalised monogamy inequalities are satisfied in this case as well.

## Introduction

Quantum entanglement is one of the most important physical resources in quantum information processing<sup>1-4</sup>. As distinguished from classical correlations, quantum entanglement cannot be freely shared among many objects. We call this important phenomenon of quantum entanglement monogamy<sup>5,6</sup>. The property of monogamy may be as fundamental as the no-cloning theorem<sup>7</sup>, which gives rise to structures of entanglement in multipartite settings<sup>8,9</sup>. Some monogamy inequalities have been studied to apply entanglement to more useful quantum information processing. The property of monogamy property has been considered in many areas of physics: it can be used to extract an estimate of the quantity of information about a secret key captured by an eavesdropper in quantum cryptography<sup>10,11</sup>, as well as the frustration effects observed in condensed matter physics<sup>12,13</sup> and even black-hole physics<sup>14,15</sup>.

The monogamy relation of entanglement is a way to characterise different types of entanglement distribution. The first monogamy relation was named the Coffman-Kundu-Wootters (CKW) inequality<sup>8</sup>. The monogamy property can be interpreted as the following statement: the amount of entanglement between  $A$  and  $B$  plus the amount of entanglement between  $A$  and  $C$  cannot be greater than the amount of entanglement between  $A$  and the  $BC$  pair. Osborne and Verstraete later proved that the CKW inequality also holds in an  $n$ -qubit system<sup>9</sup>. Other types of monogamy relations for entanglement were also proposed. Studies have found that the monogamy inequality holds in terms of some entanglement measures, negativity<sup>16</sup>, squared CREN<sup>17</sup>, entanglement of formation<sup>18-20</sup>, Rényi entropy<sup>21</sup> and Tsallis entropy<sup>22,23</sup>. The monogamy property of other physical resources, such as discord and steering<sup>24</sup>, has also been discussed. There can be several inequivalent types of entanglement among the subsystems in multipartite quantum systems, and the amount of different types of entanglement might not be directly comparable to one another. Regula *et al.* studied multi-party quantum entanglement and found that there was strong monogamy<sup>25</sup>. Additionally, generalised monogamy relations of concurrence for  $N$ -qubit systems were also proposed by Zhu *et al.*<sup>26</sup>.

In this paper, we study the generalised monogamy inequalities of CREN in multi-qubit systems. We first recall some basic concepts of entanglement measures. Then, monogamy inequalities are given by the concurrence and negativity of the  $n$ -qubit entanglement. Furthermore, we consider some states in a higher-dimensional quantum system and find that the generalised monogamy inequalities also hold for these states. We specifically test the generalised monogamy inequalities for qudits by considering the partially coherent superposition of a generalised W-class state in a vacuum, and we show that the generalised monogamy inequalities are satisfied in this case as well. These relations also give rise to a type of trade-off in inequalities that is related to the upper and lower bounds of CRENOA. It shows the bipartite entanglement between  $AB$  and the other qubits: especially under partition  $AB$ , a two-qubit system is different from the previous monogamy inequality that is typically used.

## Results

This paper is organised as follows: in the first subsection, we recall some basic concepts of concurrence and negativity. We present the monogamy relations of concurrence and negativity in the second subsection. In the third subsection, the generalised

monogamy inequalities of CREN are given. The fourth subsection includes some examples that verify these results.

## Preliminaries: concurrence and negativity

For any bipartite pure state  $|\psi\rangle_{AB}$  in a  $d \otimes d'$  ( $d \leq d'$ ) quantum system with its Schmidt decomposition,

$$|\psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |ii\rangle, \quad \lambda_i \geq 0, \quad \sum_{i=0}^{d-1} \lambda_i = 1, \quad (1)$$

the concurrence  $\mathcal{C}(|\psi\rangle_{AB})$  is defined as<sup>27</sup>

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]}, \quad (2)$$

where  $\rho_A = \text{tr}_B(|\psi\rangle_{AB}\langle\psi|)$ . For any mixed state  $\rho_{AB}$ , its concurrence is defined as

$$\mathcal{C}(\rho_{AB}) = \min \sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB}), \quad (3)$$

where the minimum is taken over all possible pure state decompositions  $\{p_i, |\psi_i\rangle_{AB}\}$  of  $\rho_{AB}$ .

Similarly, the concurrence of assistance (COA) of  $\rho_{AB}$  is defined as<sup>28</sup>

$$\mathcal{C}_a(\rho_{AB}) = \max \sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB}), \quad (4)$$

where the maximum is taken over all possible pure state decompositions  $\{p_i, |\psi_i\rangle_{AB}\}$  of  $\rho_{AB}$ .

Another well-known quantification of bipartite entanglement is negativity. For any bipartite pure state  $|\psi\rangle_{AB}$ , the negativity  $\mathcal{N}(|\psi\rangle_{AB})$  is

$$\mathcal{N}(|\psi\rangle_{AB}) = 2 \sum_{i < j} \sqrt{\lambda_i \lambda_j} = (\text{Tr} \sqrt{\rho_A})^2 - 1, \quad (5)$$

where  $\rho_A = \text{tr}_B(|\psi\rangle_{AB}\langle\psi|)$ .

For any bipartite state  $\rho_{AB}$  in the Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$  negativity is defined as<sup>29</sup>

$$\mathcal{N}(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\| - 1}{2}, \quad (6)$$

where  $\rho_{AB}^{T_A}$  is a partial transposition with respect to the subsystem A,  $\|X\|$  denotes the trace norm of  $X$ ; i.e.,  $\|X\| \equiv \text{Tr} \sqrt{XX^\dagger}$ . Negativity is a computable measure of entanglement, which is a convex function of  $\rho_{AB}$ . It disappears if, and only if,  $\rho_{AB}$  is separable for the  $2 \otimes 2$  and  $2 \otimes 3$  systems<sup>30</sup>. For the purposes of this discussion, we use the following definition of negativity:

$$\mathcal{N}(\rho_{AB}) = \|\rho_{AB}^{T_A}\| - 1. \quad (7)$$

For any maximally entangled state in a two-qubit system, this negativity is equal to 1. CREN gives a perfect discrimination of positive partial transposition-bound entangled states and separable states in any bipartite quantum system<sup>31,32</sup>. For any mixed state  $\rho_{AB}$ , CREN is defined as

$$\mathcal{N}_c(\rho_{AB}) = \min \sum_i p_i \mathcal{N}(|\psi_i\rangle_{AB}), \quad (8)$$

where the minimum is taken over all possible pure state decompositions  $\{p_i, |\psi_i\rangle_{AB}\}$  of  $\rho_{AB}$ .

For any mixed state  $\rho_{AB}$ , CRENOA is defined as<sup>17</sup>

$$\mathcal{N}_a(\rho_{AB}) = \max \sum_i p_i \mathcal{N}(|\psi_i\rangle_{AB}), \quad (9)$$

where the maximum is taken over all possible pure state decompositions  $\{p_i, |\psi_i\rangle_{AB}\}$  of  $\rho_{AB}$ .

CREN is equivalent to concurrence for any pure state with Schmidt rank-2<sup>17</sup>, and consequently, it follows that for any two-qubit mixed state  $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ :

$$\mathcal{N}_c(\rho_{AB}) = \min \sum_i p_i \mathcal{N}(|\psi_i\rangle_{AB}) = \min \sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB}) = \mathcal{C}(\rho_{AB}) \quad (10)$$

and

$$\mathcal{N}_a(\rho_{AB}) = \max \sum_i p_i \mathcal{N}(|\psi_i\rangle_{AB}) = \max \sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB}) = \mathcal{C}_a(\rho_{AB}), \quad (11)$$

where the minimum and the maximum are taken over all pure state decompositions  $\{p_i, |\psi_i\rangle_{AB}\}$  of  $\rho_{AB}$ .

## Monogamy relations of concurrence and negativity

The CKW inequality<sup>8</sup> was first defined as

$$\mathcal{C}^2(\rho_{A|BC}) \geq \mathcal{C}^2(\rho_{AB}) + \mathcal{C}^2(\rho_{AC}), \quad (12)$$

where  $\mathcal{C}(\rho_{A|BC})$  is the concurrence of a three-qubit state  $\rho_{A|BC}$  for any bipartite cut of subsystems between  $A$  and  $BC$ . Similarly, the dual inequality in terms of COA is as follows<sup>33</sup>:

$$\mathcal{C}^2(\rho_{A|BC}) \leq \mathcal{C}_a^2(\rho_{AB}) + \mathcal{C}_a^2(\rho_{AC}). \quad (13)$$

For any pure state  $|\psi\rangle_{A_1 \dots A_n}$  in an  $n$ -qubit system  $A_1 \otimes \dots \otimes A_n$ , where  $A_i \cong C^2$  for  $i = 1, \dots, n$ , a generalisation of the CKW inequality is

$$\mathcal{C}^2(|\psi\rangle_{A_1|A_2 \dots A_n}) \geq \mathcal{C}^2(\rho_{A_1 A_2}) + \dots + \mathcal{C}^2(\rho_{A_1 A_n}). \quad (14)$$

The dual inequality in terms of the COA for  $n$ -qubit states has the form<sup>17</sup>

$$\mathcal{C}^2(|\psi\rangle_{A_1|A_2 \dots A_n}) \leq \mathcal{C}_a^2(\rho_{A_1 A_2}) + \dots + \mathcal{C}_a^2(\rho_{A_1 A_n}). \quad (15)$$

When the rank of the matrix is 2, we have

$$\mathcal{C}(|\psi\rangle_{A_1|A_2 \dots A_n}) = \mathcal{N}(|\psi\rangle_{A_1|A_2 \dots A_n}). \quad (16)$$

Combining Eq. (10) with Eq. (11), we have

$$\mathcal{C}(\rho_{A_i A_j}) = \mathcal{N}_c(\rho_{A_i A_j}), \quad \mathcal{C}_a(\rho_{A_i A_j}) = \mathcal{N}_a(\rho_{A_i A_j}), \quad (17)$$

where  $i, j \in \{1, \dots, n\}, i \neq j$ .

For any  $n$ -qubit pure state  $|\psi\rangle_{A_1 \dots A_n}$ , we have

$$\mathcal{N}^2(|\psi\rangle_{A_1|A_2 \dots A_n}) \geq \mathcal{N}_c^2(\rho_{A_1 A_2}) + \dots + \mathcal{N}_c^2(\rho_{A_1 A_n}). \quad (18)$$

The dual inequality<sup>17</sup> in terms of CRENOA is as follows:

$$\mathcal{N}^2(|\psi\rangle_{A_1|A_2 \dots A_n}) \leq \mathcal{N}_a^2(\rho_{A_1 A_2}) + \dots + \mathcal{N}_a^2(\rho_{A_1 A_n}). \quad (19)$$

## Monogamy inequalities of CREN

For a  $2 \otimes 2 \otimes m$  quantum pure state  $|\psi\rangle_{ABC}$ , it has been shown that  $\mathcal{C}_a^2(\rho_{AB}) = \mathcal{C}^2(\rho_{AB}) + \tau_2^C(|\psi\rangle_{ABC})^2$ <sup>33</sup>, where  $\tau_2^C(|\psi\rangle_{ABC}) = \mathcal{C}^2(|\psi\rangle_{A|BC}) - \mathcal{C}^2(\rho_{AB}) - \mathcal{C}^2(\rho_{AC})$  is the three-tangle of concurrence.  $\mathcal{C}(|\psi\rangle_{A|BC})$  is the concurrence under bipartition  $A|BC$  for pure state  $|\psi\rangle_{ABC}$ . Namely,

$$\mathcal{C}^2(|\psi\rangle_{A|BC_1 \dots C_{n-2}}) = \mathcal{C}_a^2(\rho_{AB}) + \mathcal{C}^2(\rho_{A|C_1 \dots C_{n-2}}). \quad (20)$$

Similarly, considering that CREN is equivalent to concurrence by Eq. (17), we have

$$\mathcal{N}^2(|\psi\rangle_{A|BC_1 \dots C_{n-2}}) = \mathcal{N}_a^2(\rho_{AB}) + \mathcal{N}_c^2(\rho_{A|C_1 \dots C_{n-2}}). \quad (21)$$

The concurrence is related to the linear entropy of a state<sup>34</sup>

$$T(\rho) = 1 - \text{Tr}(\rho^2). \quad (22)$$

Given a bipartite state  $\rho$ ,  $T(\rho)$  has the property<sup>35</sup>,

$$T(\rho_A) + T(\rho_B) \geq T(\rho_{AB}) \geq |T(\rho_A) - T(\rho_B)|. \quad (23)$$

From the definition of pure state concurrence in Eq. (2) together with Eq. (22), we have

$$\mathcal{C}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) = 2[1 - \text{Tr}(\rho_{AB}^2)] = 2T(\rho_{AB}). \quad (24)$$

Now, we provide the following theorems:

**Theorem 1** For any  $2 \otimes 2 \otimes 2$  tripartite mixed state  $\rho_{ABC}$  we have

$$\mathcal{N}_a^2(\rho_{A|BC}) \leq \mathcal{N}_a^2(\rho_{B|AC}) + \mathcal{N}_a^2(\rho_{C|AB}). \quad (25)$$

*Proof.* Let  $\rho_{ABC} = \sum_i p_i |\psi_i\rangle_{ABC} \langle \psi_i|$  be an optimal decomposition realising  $\mathcal{N}_a(\rho_{A|BC})$ ; that is,

$$\mathcal{N}_a(\rho_{A|BC}) = \max_i \sum_i p_i \mathcal{N}(|\psi_i\rangle_{A|BC}), \quad (26)$$

where  $\rho_{BC} = \text{Tr}_A |\psi_i\rangle_{ABC} \langle \psi_i|$ ,  $\rho_B = \text{Tr}_{AC} |\psi_i\rangle_{ABC} \langle \psi_i|$  and  $\rho_C = \text{Tr}_{AB} |\psi_i\rangle_{ABC} \langle \psi_i|$ , and we have

$$\mathcal{N}^2(|\psi_i\rangle_{A|BC}) = \mathcal{C}^2(|\psi_i\rangle_{A|BC}) = 2T(\rho_A) = 2T(\rho_{BC}). \quad (27)$$

Combining Eq. (23) with Eq. (24), we have

$$\begin{aligned} 2T(\rho_{BC}) &\leq 2T(\rho_B) + 2T(\rho_C) \\ &= \mathcal{C}^2(|\psi_i\rangle_{B|AC}) + \mathcal{C}^2(|\psi_i\rangle_{C|AB}) \\ &= \mathcal{N}^2(|\psi_i\rangle_{B|AC}) + \mathcal{N}^2(|\psi_i\rangle_{C|AB}). \end{aligned} \quad (28)$$

The third equality holds because CRENO and concurrence are equal for any rank-2 pure state. Therefore, we obtain

$$\mathcal{N}^2(|\psi_i\rangle_{A|BC}) \leq \mathcal{N}^2(|\psi_i\rangle_{B|AC}) + \mathcal{N}^2(|\psi_i\rangle_{C|AB}). \quad (29)$$

Combining Eq. (26) with Eq. (29), we finally get

$$\mathcal{N}_a^2(\rho_{A|BC}) \leq \mathcal{N}_a^2(\rho_{B|AC}) + \mathcal{N}_a^2(\rho_{C|AB}). \quad (30)$$

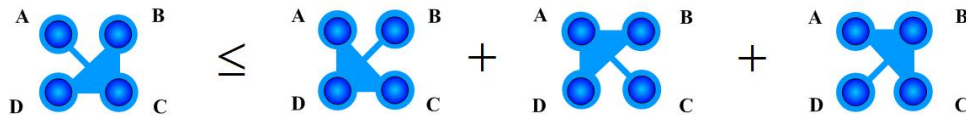
Thus, the proof is completed.  $\square$

Theorem 1 shows a simple relationship of CRENOA in a tripartite quantum system. The monogamy inequality shows that the entanglement  $A|BC$  cannot be greater than the sum of the entanglement  $B|AC$  and the entanglement  $C|AB$ . Taking an easy example, when considering a three-qubit state, the following equation exists:  $|\psi\rangle_{ABC} = a|010\rangle + b|100\rangle$  where  $|a|^2 + |b|^2 = 1$ . Using a simple calculation, the following equation can be obtained:  $\mathcal{N}_a^2(\rho_{A|BC}) = \mathcal{N}_a^2(\rho_{B|AC}) + \mathcal{N}_a^2(\rho_{C|AB})$  where the state  $|\psi\rangle_{ABC}$  saturates the monogamy inequality in Eq. (25). Moreover, the iteration of Eq. (25) leads us to the generalized monogamy inequality in multi-qubit quantum systems.

**Corollary 1** For any multi-party mixed state  $\rho_{A_1|A_2 \dots A_n}$  in an  $n$ -qubit system<sup>36</sup>, the following monogamy inequality exists:

$$\mathcal{N}_a^2(\rho_{A_1|A_2 \dots A_n}) \leq \sum_{i=2}^n \mathcal{N}_a^2(\rho_{A_i|A_1 \dots A_{i-1} A_{i+1} \dots A_n}) \leq \sum_{i=2}^n \sum_{j=1, j \neq i}^n \mathcal{N}_a^2(\rho_{A_i A_j}). \quad (31)$$

The meaning of the first inequality is clear the bipartite entanglement between  $\rho_{A_1}$  and the other qubits, when taken as a group cannot be greater than the sum of the  $n - 1$  individual bipartite entanglements between  $\rho_{A_i}$  ( $i \neq 1$ ) and the other remaining qubits. We now start to consider a four-qubit system. As shown in Fig. (a), the squared CRENOA with respect to the bipartition  $(A|BCD)$  is not greater than the sum of the three squared CRENOAs (the three possible bipartitions are  $B|ACD$ ,  $C|ABD$  and  $D|ABC$ ).



(a) The example shows the reciprocal relation of squared CRENOA in a four-qubit system.

The meaning of the second inequality is clear the sum of the bipartite entanglements between  $\rho_{A_i}$  ( $i \neq 1$ ) and the other remaining qubits cannot be greater than the sum of the bipartite entanglements  $\rho_{A_i A_j}$  ( $i \neq 1, j \neq i$ ).

**Theorem 2** For any  $n$ -qubit pure state  $|\psi\rangle_{ABC_1 \dots C_{n-2}}$ , we have

$$2\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \geq \mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}), \quad (32)$$

where  $\rho_{AB} = \text{Tr}_{C_1 \dots C_{n-2}}(|\psi\rangle\langle\psi|)$ ,  $\rho_{AC_i} = \text{Tr}_{BC_1 \dots C_{i-1}C_{i+1} \dots C_{n-2}}(|\psi\rangle\langle\psi|)$  and  $\rho_{BC_i} = \text{Tr}_{AC_1 \dots C_{i-1}C_{i+1} \dots C_{n-2}}(|\psi\rangle\langle\psi|)$ .

*Proof.* From the result of Theorem 1, we find that the generalised monogamy inequality can be easily obtained by using the superposition of states. We now consider  $\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}})$ . When the rank of the matrix is 2, we have

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) = \mathcal{C}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) = 2T(\rho_{AB}). \quad (33)$$

Combining Eq. (23) with Eq. (24), we get the relationship

$$\begin{aligned} 2T(\rho_{AB}) &\leq 2T(\rho_A) + 2T(\rho_B) \\ &= \mathcal{C}^2(|\psi_i\rangle_{A|BC_1 \dots C_{n-2}}) + \mathcal{C}^2(|\psi_i\rangle_{B|AC_1 \dots C_{n-2}}) \\ &= \mathcal{N}^2(|\psi_i\rangle_{A|BC_1 \dots C_{n-2}}) + \mathcal{N}^2(|\psi_i\rangle_{B|AC_1 \dots C_{n-2}}). \end{aligned} \quad (34)$$

The third equality follows from the fact that CREN and concurrence are equal for any rank-2 pure state.

$$\mathcal{N}^2(|\psi_i\rangle_{AB|C_1 \dots C_{n-2}}) \leq \mathcal{N}^2(|\psi_i\rangle_{A|BC_1 \dots C_{n-2}}) + \mathcal{N}^2(|\psi_i\rangle_{B|AC_1 \dots C_{n-2}}). \quad (35)$$

For a mixed state, CRENOA is expressed as  $\mathcal{N}(|\psi_i\rangle_{A|BC_1 \dots C_{n-2}})$ , and we have

$$\mathcal{N}(\rho_{A|BC_1 \dots C_{n-2}}) = \max_i \sum_i p_i \mathcal{N}(|\psi_i\rangle_{A|BC_1 \dots C_{n-2}}). \quad (36)$$

Furthermore, when combining this with Eq. (35), we finally get

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) \leq \mathcal{N}_a^2(\rho_{A|BC_1 \dots C_{n-2}}) + \mathcal{N}_a^2(\rho_{B|AC_1 \dots C_{n-2}}) \quad (37)$$

and

$$\begin{aligned} \mathcal{N}_a^2(\rho_{A|BC_1 \dots C_{n-2}}) &\leq \mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}), \\ \mathcal{N}_a^2(\rho_{B|AC_1 \dots C_{n-2}}) &\leq \mathcal{N}_a^2(\rho_{BA}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}). \end{aligned} \quad (38)$$

Combining Eq. (37) with Eq. (38), we have Eq. (32). In other words, we give an upper bound about  $\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}})$ , i.e.,

$$2\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \geq \mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}). \quad (39)$$

This completes the proof.  $\square$

Theorem 2 shows that the entanglement between  $AB$  and the other qubits cannot be greater than the sum of the individual entanglements between  $A$  and each of the  $n-1$  remaining qubits and the individual entanglements between  $B$  and each of the  $n-1$  remaining qubits. Theorem 2 provides a polygamy-type upper bound of multi-qubit entanglement between the two-qubit system  $AB$  and the other  $(n-2)$ -qubit system  $C_1 C_2 \dots C_{n-2}$  in terms of the squared CRENOA. Especially under partition  $AB$ , a two-qubit system is different from the previous monogamy inequality. When  $|\psi\rangle_{AB|C_1 \dots C_{n-2}} = |\psi\rangle_A \otimes |\psi\rangle_{B|C_1 \dots C_{n-2}}$ , the calculation results in  $\mathcal{N}_a^2(\rho_{AB}) = 0$ ,  $\mathcal{N}_a^2(\rho_{AC_i}) = 0$ . Consequently, the polygamy-type relation is obtained as shown in Eq. (19).

Finally, consider the following four-qubit state:  $|\psi\rangle_{ABCD} = a|0100\rangle + b|0010\rangle + c|0001\rangle$  where  $|a|^2 + |b|^2 + |c|^2 = 1$ . We can easily get the following equations:  $\mathcal{N}_a^2(\rho_{AB}) = \mathcal{N}_a^2(\rho_{AC}) = \mathcal{N}_a^2(\rho_{AD}) = 0$  and  $\mathcal{N}_a^2(\rho_{BC}) + \mathcal{N}_a^2(\rho_{BD}) = \mathcal{N}^2(|\psi\rangle_{AB|CD}) = \frac{16}{9}$ . Therefore, the state  $|\psi\rangle_{ABCD}$  saturates the monogamy inequality in Eq. (32).

**Theorem 3** For any  $n$ -qubit pure state  $|\psi\rangle_{ABC_1 \dots C_{n-2}}$ ,

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) \geq \left| \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right|, \quad (40)$$

where  $\rho_{AB} = \text{Tr}_{C_1 \dots C_{n-2}}(|\psi\rangle\langle\psi|)$ ,  $\rho_{AC_i} = \text{Tr}_{BC_1 \dots C_{i-1}C_{i+1} \dots C_{n-2}}(|\psi\rangle\langle\psi|)$  and  $\rho_{BC_i} = \text{Tr}_{AC_1 \dots C_{i-1}C_{i+1} \dots C_{n-2}}(|\psi\rangle\langle\psi|)$ .

*Proof.* We have the following property for linear entropy<sup>35</sup>:

$$T(\rho_{AB}) \geq |T(\rho_A) - T(\rho_B)|. \quad (41)$$

Combining Eq. (24) with Eq. (41), we have

$$2[1 - \text{Tr}(\rho_{AB}^2)] \geq |2[1 - \text{Tr}(\rho_A^2)] - 2[1 - \text{Tr}(\rho_B^2)]| \quad (42)$$

and

$$\mathcal{C}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) \geq |\mathcal{C}^2(|\psi\rangle_{A|BC_1 \dots C_{n-2}}) - \mathcal{C}^2(|\psi\rangle_{B|AC_1 \dots C_{n-2}})|. \quad (43)$$

By using the equivalent relation between concurrence and CREN (see Eq. (17)), we have

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) \geq |\mathcal{N}^2(|\psi\rangle_{A|BC_1 \dots C_{n-2}}) - \mathcal{N}^2(|\psi\rangle_{B|AC_1 \dots C_{n-2}})|. \quad (44)$$

There is a relationship between CREN and CRENOA (see Eq. (21)):

$$\mathcal{N}^2(|\psi\rangle_{A|BC_1 \dots C_{n-2}}) = \mathcal{N}_a^2(\rho_{AB}) + \mathcal{N}_c^2(\rho_{A|C_1 \dots C_{n-2}}) \quad (45)$$

$$\mathcal{N}^2(|\psi\rangle_{B|AC_1 \dots C_{n-2}}) = \mathcal{N}_a^2(\rho_{BA}) + \mathcal{N}_c^2(\rho_{B|C_1 \dots C_{n-2}}). \quad (46)$$

Putting the above two equalities into Eq. (44), we get

$$|\mathcal{N}_c^2(\rho_{A|C_1 \dots C_{n-2}}) - \mathcal{N}_c^2(\rho_{B|C_1 \dots C_{n-2}})| \geq \left| \sum_{i=1}^{n-2} \mathcal{N}_c^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right| \geq \left| \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right|. \quad (47)$$

Similar to the above derivation, we give a lower bound about  $\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}})$ , i.e.,

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1 \dots C_{n-2}}) \geq \left| \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right|. \quad (48)$$

This lower bound is a direct consequence of CREN.  $\square$

Theorem 3 shows that the entanglement between  $AB$  and the other qubits cannot be less than the absolute value of the difference between both the individual entanglements between  $A$  and each of the  $n-1$  remaining qubits and the individual entanglements between  $B$  and each of the  $n-1$  remaining qubits. Theorem 3 provides a monogamy-type lower bound of multi-qubit entanglement between the two-qubit system  $AB$  and the other  $(n-2)$ -qubit system  $C_1 C_2 \dots C_{n-2}$  in terms of the squared CRENOA. When  $|\psi\rangle_{AB|C_1 \dots C_{n-2}} = |\psi\rangle_B \otimes |\psi\rangle_{A|C_1 \dots C_{n-2}}$ ,  $\mathcal{N}_a^2(\rho_{BC_i}) = 0$ , and so we obtain the CWK-type relation in Eq. (18).

Finally, we consider the following four-qubit state  $|\psi\rangle_{ABCD} = a|1000\rangle + b|0010\rangle + c|0001\rangle$  where  $|a|^2 + |b|^2 + |c|^2 = 1$ , from which we can easily obtain the following equations:  $\mathcal{N}_a^2(\rho_{BC}) = \mathcal{N}_a^2(\rho_{BD}) = 0$  and  $\mathcal{N}_a^2(\rho_{AC}) + \mathcal{N}_a^2(\rho_{AD}) = \mathcal{N}^2(|\psi\rangle_{AB|CD}) = \frac{16}{9}$ . Therefore, the state  $|\psi\rangle_{ABCD}$  saturates the monogamy inequality in Eq. (40). Therefore, a generalised monogamy inequality using negativity and CRENOA in an  $n$ -qubit is proposed. These relations also give rise to a type of trade-off in inequalities that is related to the upper and lower bounds of CRENOA.

**Remark** It is interesting to note that the properties of CREN are based on the subadditivity of linear entropy. However, negativity violates this subadditivity in general conditions<sup>37–39</sup>.

## Examples

In this section, we use some special states to study generalised monogamy inequalities. First, we consider the (Greenberger-Horne-Zeilinger) GHZ state and W state in Examples 1 and 2. Second, we consider two states in the higher-dimensional system in Examples 3 and 4.

**Example 1.** For an arbitrary pure GHZ state in an  $n$ -qubit system:

$$|GHZ\rangle = a|0\rangle^{\otimes n} + b|1\rangle^{\otimes n}, \quad (49)$$

where  $|a|^2 + |b|^2 = 1$ . The generalized GHZ state is satisfied with the previous CKW inequality. We will now show that the generalised GHZ state satisfies the generalised monogamy inequalities. We have  $\rho_1 = \rho_2 = \dots = \rho_n = a^2|0\rangle\langle 0| + b^2|1\rangle\langle 1|$ . It is straightforward to check:  $\mathcal{N}_a^2(\rho_{A_1|A_2 \dots A_n}) = \mathcal{N}_a^2(\rho_{A_2|A_1 \dots A_n}) = \dots = \mathcal{N}_a^2(\rho_{A_i|A_1 \dots A_{i-1} A_{i+1} \dots A_n}) = 4|(ab)^2|$  and  $\mathcal{N}_a^2(\rho_{A_1 A_2}) = \mathcal{N}_a^2(\rho_{A_2 A_3}) = \dots = \mathcal{N}_a^2(\rho_{A_i A_{i+1}}) = 4|ab|$ ,  $\mathcal{N}^2(\rho_{A_1 A_2 | A_3 \dots A_n}) = 4|(ab)^2|$ . Therefore:

$$\mathcal{N}_a^2(\rho_{A_1 | A_2 \dots A_n}) = 4|(ab)^2| \leq 4(n-1)|(ab)^2| = \sum_{i=2}^n \mathcal{N}_a^2(\rho_{A_i | A_1 \dots A_{i-1} A_{i+1} \dots A_n}), \quad (50)$$

$$2\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) = 8(n-1)|(ab)^2| \geq 4|(ab)^2| = \mathcal{N}^2(|\psi\rangle_{AB|C_1\dots C_{n-2}}), \quad (51)$$

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1\dots C_{n-2}}) = 4|(ab)^2| \geq 0 = \left| \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right|. \quad (52)$$

Example 2. For a pure state  $|W\rangle$  in an  $n$ -qubit system:

$$|W\rangle = \frac{1}{\sqrt{n}}(|10\dots 0\rangle + |01\dots 0\rangle + \dots + |00\dots 1\rangle), \quad (53)$$

with  $\sum_{i=1}^n |\frac{1}{\sqrt{n}}|^2 = 1$ . It is very important to understand the saturation of the previous CKW inequality. Using a simple calculation, we have  $\rho_1 = \rho_2 = \dots = \rho_n = \frac{1}{n}(|1\rangle\langle 1| + \frac{n-1}{n}(|0\rangle\langle 0|)$ . It is straightforward to check:  $\mathcal{N}_a^2(\rho_{A_1|A_2\dots A_n}) = \mathcal{N}_a^2(\rho_{A_2|A_1\dots A_n}) = \dots = \mathcal{N}_a^2(\rho_{A_i|A_1\dots A_{i-1}A_{i+1}\dots A_n}) = \frac{4(n-1)}{n^2}$ .  $\mathcal{N}_a^2(\rho_{A_1A_2}) = \mathcal{N}_a^2(\rho_{A_2A_3}) = \dots = \mathcal{N}_a^2(\rho_{A_iA_{i+1}}) = \frac{4}{n^2}$ ,  $\mathcal{N}^2(\rho_{A_1A_2|A_3\dots A_n}) = \frac{8(n-2)}{n^2}$ . In the same way, we get the following inequalities:

$$\mathcal{N}_a^2(\rho_{A_1|A_2\dots A_n}) = \frac{4(n-1)}{n^2} \leq \frac{4(n-1)^2}{n^2} = \sum_{i=2}^n \mathcal{N}_a^2(\rho_{A_i|A_1\dots A_{i-1}A_{i+1}\dots A_n}), \quad (54)$$

$$2\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) = \frac{8(n-1)}{n^2} \geq \frac{8(n-2)}{n^2} = \mathcal{N}^2(|\psi\rangle_{AB|C_1\dots C_{n-2}}), \quad (55)$$

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1\dots C_{n-2}}) = \frac{8(n-2)}{n^2} \geq 0 = \left| \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right|. \quad (56)$$

From the above results, we discover that the generalised GHZ state and W state satisfy our inequalities. We further explore the condition of the generalised inequalities in higher-dimensional systems. We consider the following examples:

Example 3. For a pure, totally antisymmetric state  $|\psi_{ABC}\rangle$  in a  $3 \otimes 3 \otimes 3$  system<sup>40</sup>:

$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle). \quad (57)$$

This special quantum state is not satisfied with the previous CKW inequality<sup>41</sup> but it is established in generalised monogamy inequalities. We can easily obtain  $\mathcal{N}_a^2(\rho_{A|BC}) = \mathcal{N}_a^2(\rho_{B|AC}) = \mathcal{N}_a^2(\rho_{C|AB}) = 4$  and further obtain the inequalities  $\mathcal{N}_a^2(\rho_{A|BC}) \leq \mathcal{N}_a^2(\rho_{B|AC}) + \mathcal{N}_a^2(\rho_{C|AB})$ . We now explore theorems 2 and 3. First, we have  $\mathcal{N}_a^2(\rho_{AB}) = 1$ ,  $\mathcal{N}_a^2(\rho_{AC}) = 1$ ,  $\mathcal{N}_a^2(\rho_{BC}) = 1$  and  $\mathcal{N}^2(|\psi\rangle_{AB|C}) = 4$ . Therefore, we obtain the following inequalities:

$$2\mathcal{N}_a^2(\rho_{AB}) + \mathcal{N}_a^2(\rho_{AC}) + \mathcal{N}_a^2(\rho_{BC}) \geq \mathcal{N}^2(|\psi\rangle_{AB|C}) \geq |\mathcal{N}_a^2(\rho_{AC}) - \mathcal{N}_a^2(\rho_{BC})|. \quad (58)$$

Example 4. The  $n$ -qudit generalised W-class state in higher-dimensional quantum systems is very useful in quantum information theory<sup>42</sup>. We verify whether the generalised monogamy inequalities hold in higher-dimensional systems using a special example. First, we recall the definition of  $n$ -qudit generalised W-class state<sup>43</sup>,

$$|W_n^d\rangle_{A_1\dots A_n} = \sum_{i=1}^{d-1} (a_{1i}|i0\dots 0\rangle + a_{2i}|0i\dots 0\rangle + \dots + a_{ni}|00\dots i\rangle), \quad (59)$$

where  $\sum_{s=1}^n \sum_{i=1}^{d-1} |a_{si}|^2 = 1$ .

Let  $|\psi\rangle_{A_1\dots A_n}$  be an  $n$ -qudit pure state in a superposition of an  $n$ -qudit generalised W-class state and vacuum; that is,

$$|\psi\rangle_{A_1\dots A_n} = \sqrt{p}|W_n^d\rangle_{A_1\dots A_n} + \sqrt{1-p}|0\dots 0\rangle_{A_1\dots A_n}, \quad (60)$$

for some  $0 \leq p \leq 1$ .



For the squared negativity  $\mathcal{N}^2$  of  $|\psi\rangle_{A_1 \dots A_n}$  with respect to the bipartition between  $A_1$  and the other qudits, the reduced density matrix  $\rho_{A_1}$  of  $|\psi\rangle_{A_1 \dots A_n}$  onto subsystem  $A_1$  is obtained as

$$\begin{aligned}\rho_{A_1} &= \text{Tr}_{A_2 \dots A_n} |\psi\rangle_{A_1 A_2 \dots A_n} \langle \psi| \\ &= p \sum_{i,j=1}^{d-1} a_{1i} a_{1j}^* |i\rangle_{A_1} \langle j| + [p\Omega + (1-p)] |0\rangle_{A_1} \langle 0| + \sqrt{p(1-p)} \left[ \sum_{i=1}^{d-1} a_{1i} |i\rangle_{A_1} \langle 0| + \sum_{j=1}^{d-1} a_{1j}^* |0\rangle_{A_1} \langle j| \right],\end{aligned}\quad (61)$$

where  $\Omega = \sum_{s=2}^n \sum_{i=1}^{d-1} |a_{si}|^2 = 1 - \sum_{j=1}^{d-1} |a_{1j}|^2$ .

When considering the  $|\psi\rangle_{A_1 A_2 \dots A_n}$  state, we need to obtain the eigenvalue of the matrix by applying the definition of pure state negativity in Eq. (5). Using a simple calculation, we find that the matrix has rank-2 and we have

$$\mathcal{N}^2(|\psi\rangle_{A_1 A_2 \dots A_n}) = [(\text{Tr} \sqrt{\rho_{A_1}})^2 - 1]^2 = 4\lambda_i \lambda_j = 4p^2(1 - \Omega)\Omega. \quad (62)$$

We now consider the case in which  $n = 2$ . The remaining cases follow analogously. The two-qudit reduced density matrix  $\rho_{A_1 A_2}$  of  $|\psi\rangle_{A_1 A_2 \dots A_n}$  is obtained as

$$\begin{aligned}\rho_{A_1 A_2} &= \text{Tr}_{A_3 \dots A_n} |\psi\rangle_{A_1 A_2 \dots A_n} \langle \psi| \\ &= p \sum_{i,j=1}^{d-1} [a_{1i} a_{1j}^* |i0\rangle_{A_1 A_2} \langle j0| + a_{1i} a_{2j}^* |i0\rangle_{A_1 A_2} \langle 0j| + a_{2i} a_{1j}^* |0i\rangle_{A_1 A_2} \langle j0| + a_{2i} a_{2j}^* |0i\rangle_{A_1 A_2} \langle 0j|] \\ &\quad + (p\Omega_2 + 1 - p) |00\rangle_{A_1 A_2} \langle 00| + \sqrt{p(1-p)} \sum_{k=1}^{d-1} [(a_{1k} |k0\rangle + a_{2k} |0k\rangle)_{A_1 A_2} \langle 00| + |00\rangle_{A_1 A_2} (a_{1k}^* \langle k0| + a_{2k}^* \langle 0k|)],\end{aligned}\quad (63)$$

where  $\Omega_2 = 1 - \sum_{j=1}^{d-1} (a_{1j}^2 + a_{2j}^2)$ . For convenient calculation, we consider two unnormalised states:

$$|\tilde{x}\rangle = \sqrt{p} \sum_{i=1}^{d-1} (a_{1i} |i0\rangle_{A_1 A_2} + a_{2i} |0i\rangle_{A_1 A_2}) + \sqrt{1-p} |00\rangle_{A_1 A_2}, \quad |\tilde{y}\rangle = \sqrt{\Omega_2} |00\rangle_{A_1 A_2}. \quad (64)$$

Consequently,  $\rho_{A_1 A_2}$  can be represented as  $\rho_{A_1 A_2} = |\tilde{x}\rangle_{A_1 A_2} \langle \tilde{x}| + |\tilde{y}\rangle_{A_1 A_2} \langle \tilde{y}|$ , where  $|\tilde{x}\rangle$  and  $|\tilde{y}\rangle$  are unnormalised states of the subsystems  $A_1 A_2$ . By the HJW theorem<sup>44</sup>, any pure-state decomposition  $\rho_{A_1 A_2} = \sum_h^r |\tilde{\psi}_h\rangle_{A_1 A_2} \langle \tilde{\psi}_h|$ , with size  $r > 2$  can be obtained by an  $r \times r$  unitary matrix  $u_{hl}$  such that

$$|\tilde{\psi}_h\rangle_{A_1 A_2} = u_{h1} |\tilde{x}\rangle_{A_1 A_2} + u_{h2} |\tilde{y}\rangle_{A_1 A_2} \quad (65)$$

for each  $h$ , for the normalized state  $|\psi_h\rangle_{A_1 A_2} = |\tilde{\psi}_h\rangle_{A_1 A_2} / \sqrt{p_h}$  with  $p_h = |\langle \tilde{\psi}_h | \tilde{\psi}_h \rangle|$ .

We apply the definition of mixed state negativity in Eq. (8) and Eq. (63), and then we have the two-tangle based on the CREN of  $\rho_{A_1 A_2}$  as

$$\mathcal{N}_c^2(\rho_{A_1 A_2}) = \min_i \sum_i p_i \mathcal{N}^2(|\psi_i\rangle_{A_1 A_2}) = 4p^2(1 - \Omega) \sum_{i=1}^{d-1} |a_{2i}|^2 = 4p^2(1 - \Omega)\Omega', \quad (66)$$

where  $\Omega' = \sum_{i=1}^{d-1} |a_{2i}|^2$ .

From the definition of pure state negativity in Eq. (9) and Eq. (63), we have

$$\mathcal{N}_c^2(\rho_{A_1 A_2 | A_3 \dots A_n}) = \min_i \sum_i p_i \mathcal{N}^2(|\psi_i\rangle_{A_1 A_2 | A_3 \dots A_n}) = 4p\Omega_2(1 - \Omega_2). \quad (67)$$

We now try to verify the generalised monogamy inequalities of CREN in an  $n$ -qudit system. For convenient calculation, we assume that  $\sum_{i=1}^{d-1} a_{1i}^2 = a$ ,  $\sum_{i=1}^{d-1} a_{2i}^2 = b$ ,  $\sum_{i=1}^{d-1} a_{1i}^4 = A$ ,  $\sum_{i=1}^{d-1} a_{2i}^4 = B$ .

We first consider the generalisation of Theorem 1.

$$\mathcal{N}_a^2(\rho_{A_1 | A_2 \dots A_n}) = \mathcal{N}_a^2(\rho_{A_2 | A_1 \dots A_n}) = \dots = \mathcal{N}_a^2(\rho_{A_i | A_1 \dots A_{i-1} A_{i+1} \dots A_n}) = 4p^2(1 - \Omega)\Omega = 4p^2(1 - a)a. \quad (68)$$

This special quantum state is satisfied with the generalised monogamy inequality in Eq. (25) i.e.,

$$\mathcal{N}_a^2(\rho_{A_1 | A_2 \dots A_n}) \leq \sum_{i=2}^n \mathcal{N}_a^2(\rho_{A_i | A_1 \dots A_{i-1} A_{i+1} \dots A_n}). \quad (69)$$



For the generalisation of Theorem 2, the left of Eq. (32) is

$$2\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}). \quad (70)$$

Using Eq. (8) and Eq. (62) we can simplify the calculation to

$$\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) = \mathcal{N}_c^2(\rho_{A|BC_1\dots C_n}) = 4p^2(1-\Omega)\Omega = 4p^2(1-a)a \quad (71)$$

and

$$\sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) = \mathcal{N}_c^2(\rho_{B|C_1\dots C_n}) = 4p^2(1-\Omega')\Omega' = 4p^2(1-b)b. \quad (72)$$

After some calculations, we have

$$\begin{aligned} 2\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) &= \mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) + \mathcal{N}_a^2(\rho_{AB}) \\ &= \mathcal{N}_c^2(\rho_{A|BC_1\dots C_n}) + \mathcal{N}_c^2(\rho_{B|C_1\dots C_n}) + \mathcal{N}_a^2(\rho_{AB}) = 4p^2(1-a)a + 4p^2(1-b)b + 4p^2ab. \end{aligned} \quad (73)$$

Second, taking Eq. (67) to the right side of Eq. (32), we then have

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1\dots C_{n-2}}) = 4p\Omega_2(1-\Omega_2) = 4p^2[1-(a+b)](a+b). \quad (74)$$

After a straightforward calculation, we obtain

$$2\mathcal{N}_a^2(\rho_{AB}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) + \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) - \mathcal{N}^2(|\psi\rangle_{AB|C_1\dots C_{n-2}}) = 12p^2ab \geq 0. \quad (75)$$

Therefore, this  $n$ -qudit pure state is satisfied with the generalised monogamy inequality in Eq. (32). In other words, the test of the Theorem 2 has been accomplished. Next, we verify Theorem 3. First, we consider the term CREN from Eq. (40):

$$\sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) = \mathcal{N}_c^2(\rho_{A|BC_1\dots C_n}) - \mathcal{N}_a^2(\rho_{AB}) = 4p^2(1-\Omega)\Omega - 4p^2(1-\Omega)\Omega' = 4p^2a(1-a-b). \quad (76)$$

Calculating the absolute value of the difference between Eq. (72) and Eq. (76), we obtain

$$\left| \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right| = |4p^2(a-a^2-ab+b^2-b)| = 4p^2(a-a^2-ab+b^2-b). \quad (77)$$

It is easy to check  $4p^2(a-a^2-ab+b^2-b) > 0$ , as

$$\begin{aligned} 0 \leq a+b \leq 1 &\Rightarrow a(a+b) \leq a \Rightarrow a^2+ab-b+b^2 < a(a+b) \leq a \Rightarrow a^2+ab-b+b^2-a < 0 \\ &\Rightarrow a-a^2-ab+b^2-b > 0 \Rightarrow 4p^2(a-a^2-ab+b^2-b) > 0. \end{aligned} \quad (78)$$

After a straightforward calculation, we have

$$\mathcal{N}^2(|\psi\rangle_{AB|C_1\dots C_{n-2}}) - \left| \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{AC_i}) - \sum_{i=1}^{n-2} \mathcal{N}_a^2(\rho_{BC_i}) \right| = 4p^2b(2-2b-a) \geq 0. \quad (79)$$

Therefore, this  $n$ -qudit pure state satisfies the generalised monogamy inequality in Eq. (40). We have now verified the generalised monogamy inequalities. In other words, the generalised monogamy inequality are satisfied with the  $n$ -qudit pure state for all three of our theorems.

## Conclusions

In this paper, we have used CREN to study different types of monogamy relations. In particular, we have shown that CREN satisfies the generalised monogamy inequalities. We have investigated the CKW-like inequalities and generalised monogamy inequalities. Furthermore, the generalised monogamy inequalities related to CREN and CRENOA were obtained by  $n$ -qubit states. These relations also give rise to a type of trade-off in inequalities that is related to the upper and lower bounds of CRENOA. Finally, we have shown that the partially coherent superposition of the generalised W-class state and vacuum extensions of CREN satisfies the generalised monogamy inequalities. We believe that the generalised monogamy inequalities can be useful in quantum information theory. This paper was based on the linear entropy. To continue this work, we will study the nature of other entropy further in the future work. We hope that our work will be useful to the quantum physics.

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## Acknowledgements

It is a pleasure to thank F.G.Zhang for inspiring discussions. We thank the anonymous referees for their valuable comments. This work was supported by the National Nature Science Foundation of China (Grant No.1127123), the Higher School Doctoral Subject Foundation of Ministry of Education of China (Grant No. 20130202110001) and the Fundamental Research Funds for the Central Universitie (Grant No. 2016CBY003).

## Author contributions statement

T.T. and Y.Luo contributed the idea. T.T. performed the calculations and wrote the main manuscript. Y.Luo checked the calculations. Y.Li improved the manuscript. All authors contributed to the discussion and reviewed the manuscript.

## Additional information

Competing financial interests: The authors declare no competing financial interests.